UDC 62-50

OPTIMIZING FUNCTIONAL IN COMBINED CONTROLS OF DYNAMIC PLANTS

PMM Vol. 36, №1, 1972, pp. 24-32

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(Received July 1, 1971)

An optimizing functional (optimality criterion) of a unified structure is found for combined controls of dynamic plants, which obey a complex set of engineering specifications. An analytic method is presented for the construction of the switching boundary. One of the possible ways of taking into account all the manifold specifications required of the performance of the motion of dynamic plants of various nature leads to the use of combined control [1]. Here, by a rational combined control is meant a control which ensures the optimality of one specific quality from the complex set of engineering specifications in a specific region of the phase space, the optimality of another quality is achieved in the next region, etc. The collection of constraints to be taken into account during the optimization is admissible in any of the regions. When the regions of optimality of each of the specific qualities are invariant, such a combined control corresponds to the particular case when the complex performance index of the control is representable in the form of a weighted sum of the partial criteria with piecewise constant weighting coefficients [2]. The fundamental problems arising in the realization of the rational combined controls of dynamic plants are the representation of the performance indices (of the functionals to be optimized) and of the controls in a single structural form. Another such problem is the choice of the switching boundary between the optimization regions of the various qualities. The aim of the present paper is to solve these problems in the special case of a dynamic plant whose motion is described by linear differential equations. A stabilization system is synthesized in the form of a rational combined control ensuring the fastest possible damping of the transient response under large perturbations, as well as high accuracy and small sensitivity to variations of the parameters of the plant and of the stabilization system under small deviations from the stable equilibrium position (which is taken to coincide with the origin of the system's phase coordinates). Under such general requirements it is necessary to choose a simple constructive representative performance index of the combined control, which is critical for the parameters being investigated, and to choose a control law of unified structure.

Let the motion of a dynamic plant be completely described by a controlled linear system which, without loss of generality, can be replaced by the system [3]

$$q^{\cdot} = \Phi q + Iu \qquad (|u| \leqslant 1) \tag{1}$$

Here q is the plant's n-dimensional phase coordinate vector, Φ is the $n \times n$

Frobenius matrix, $I' = (0,0, \ldots, 0, 1)$; the prime denotes transposition. We assume that u is a rational combined control, i.e., in a certain region Q_1 including the origin there is fulfilled the requirement of optimality with respect to accuracy and small sensitivity, while in another region Q_2 (external in relation to Q_1), the requirement of time optimality.

The analysis of relay control systems of plants and of control systems with variable structure shows [4, 5] that the requirements of accuracy and of the insensitivity of the transient response to variations of plant and control parameters can be satisfied if we require that in region Q_1 the motion of system (1) take place in a sliding mode on a certain hypersurface S without the representative point going outside the boundary of region Q_1. We treat the case when S is a hyperplane, i.e.,

$$S = \sum_{i=1}^{n} k_i q_i = 0 \tag{2}$$

The control which ensures the system a sliding mode on S has the form [4, 5]

$$u = - \operatorname{sign} S \tag{3}$$

It can be shown that to a stable sliding mode of system (1) with a control (3) there corresponds an optimality criterion of the form

$$J_1 = \int_0^\infty q' M_1 q dt \tag{4}$$

Here M_1 is an $n \times n$ symmetric positive-definite matrix. The elements of matrix M_1 are determined from the solution of the inverse optimization problem (see Appendix 1) which has the form

$$\sum_{i=1}^{n} (-1)^{i+p} \left(\frac{m^{(1)}_{i2p-i}}{m^{(1)}_{11}} - \frac{k^{(1)}_i k^{(1)}_{2p-i}}{k^{(1)2}_1} \right) = 0 \quad (p = 1, \ldots, n) \tag{5}$$

Choosing a control of form (3) satisfies also the simplicity requirement of the corresponding optimality criterion for region Q_1 since the coefficients $k_i^{(1)}$ $(i = 1, \ldots, n)$ of the optimal control are determined from a limit system of Riccati algebraic equations.

The determination of the boundaries of region Q_1 is one of the fundamental and complicated problems when combining optimal controls. These boundaries may be formed by the limit trajectories of system (1), which terminate on the boundary segments of hyperplane S, defining the sliding mode region. They may be constructed also by means of integrating system (1) in reverse time under initial conditions belonging to the boundary segments and with $u = \pm 1$, as well as by means of simulation on an electronic digital computer [1]. However, boundaries of region Q_1 determined in such fashion are described by expressions which are awkward for the analysis and synthesis of optimal systems.

In Appendix 2, on the basis of Farkas' theorem [6], it is shown that if region Q_1 is chosen in the form

$$\left| \sum_{j=1}^{n} b_{ij} q_j \right| \leqslant d_i, \quad d_i > 0 \qquad (i = 1, \ldots, r) \tag{6}$$

then the following conditions are fulfilled when $r = n$.

1. Region Q_1 is bounded if the relations

$$(\Lambda_s^{(1)} - \Lambda_s^{(2)})' B = I^{(s)}, \quad (\Lambda_s^{(1)} + \Lambda_s^{(2)})' l \geqslant 0 \quad (s = 1, \ldots, n) \tag{7}$$
$$l = (1, 1, \ldots 1), \; I^{(s)} = (\delta_{1s}, \delta_{2s}, \ldots, \delta_{ns}), \quad \delta_{is} = \begin{cases} 0, & i \neq s \\ 1, & i = s \end{cases}$$

hold, where $\Lambda_s^{(i)}$ $(i = 1,2)$ is a nonnegative n-dimensional vector.

2. After falling into region Q_1 the phase point does not go outside its boundary if the relations

$$\sum_{i=1,\, i\neq s}^{p} [l_{is}^{(1)}(-d_i + d_s) + l_{is}^{(2)}(-d_i - d_s)] \mp l_s k_n d_s \leqslant -1 + (a_n - b_{sn-1}) d_s$$
$$\sum_{i=1,\, i\neq s}^{n} (l_{is}^{(1)} - l_{is}^{(2)})(b_{ij} - b_{sj}) + l_s(k_n b_{sj} - k_j) = \mp a_j - b_{sj-1} - (a_n - b_{sn-1})\, b_{sj}$$
$$l_s, l_{is}^{(1)}, l_{is}^{(2)} \geqslant 0 \quad (i, s = 1, \ldots, n;\; i \neq s;\; j = 1, \ldots, n-1) \tag{8}$$

are fulfilled.

3. If the relations

$$\lambda\left(b_{1i} - b_{1n}\frac{k_i}{k_n}\right) = k_{i-1} - a_i k_n - \frac{k_{n-1} - a_n k_n}{k_n} k_i \quad (i = 1, \ldots, n-1) \tag{9}$$
$$-\lambda d_1 \leqslant k_n \qquad (\lambda \geqslant 0)$$

hold, a sliding mode is observed when the phase point falls into S in the region Q_1.

We now consider the region Q_2. Its outer boundaries are usually determined from constructive and technical considerations. Its inner boundaries are the boundaries of region Q_1. Therefore, the basic problem reduces to the determination of an optimization criterion whose structure is close to (4). It is also important to establish the connection of the parameters of such a criterion and of the optimal control. As was noted in [7] good results on time optimality can be achieved if the form

$$J_3 = \int_0^\infty e^{2\delta t}(q' M_3 q + cu^2)\, dt \quad (\delta > 0) \tag{10}$$

is taken as the optimization functional. Here M_3 is an $n \times n$ symmetric positive-definite matrix. The control for system (1), optimal in the sense of criterion (10), is determined as a result of solving the problem of the analytic design of controllers and has the form

$$u = \operatorname{sat} u_* = \begin{cases} u_*, & |u_*| < 1, \\ \operatorname{sign} u_*, & |u_*| \geqslant 1, \end{cases} \qquad u_* = \sum_{i=1}^{n} k_i^{(2)} q_i \tag{11}$$

For a system (1) closed by the given control (11) we can find an optimality criterion of the form [8, 9]

$$J_2 = \int_0^\infty (q' M_2 q + cu^2)\, dt \tag{12}$$

Here M_2 is an $n \times n$ symmetric positive-definite matrix. The coefficients of matrix M_2 are determined from the solution of the inverse problem for system (1) with control (11), which has the form (see Appnedix 1)

$$\sum_{i=1}^{n+1} (-1)^{i+p} (m_{i2p-1}^{(2)} c^{-1} + a_i k_{2p-i}^{(2)} + a_{2p-i} k_i^{(2)} - k_i^{(2)} k_{2p-i}^{(2)}) = 0 \quad (p = 1, \ldots, n) \qquad (13)$$

$$a_{n+1} = 1, \qquad k_{n+1} = 0, \qquad m_{in+1}^{(2)} = 0$$

Using the results of [10] we can show that the optimality regions of the closed system (1), (11) in the sense of criterion (10) and of criterion (12) coincide. The boundaries of this region D are determined by the manifold of tangent trajectories to the hyperplanes

$$\sum_{i=1}^{n} k_i^{(2)} q_i = \pm 1 \qquad (14)$$

Suppose that region Q_2 belongs to region D. Otherwise we shall consider control (11) as quasi-optimal in the sense of criterion (10) and of the corresponding criterion (12). A control of form (11) acts only in region Q_2. On reaching the boundaries of region Q_1 at an instant τ a switching of the control takes place and it has the form (3). The optimality criterion J_Σ of the plant's motion in both regions can be written as

$$J_\Sigma = \int_0^\tau [q' M_2 q + cu^2]\, dt + \int_\tau^\infty q' M_1 q dt = \int_0^\infty [q' M(q)\, q + c(q)\, u^2]\, dt \qquad (15)$$

$$\{M(q), c(q)\} = \begin{cases} \{M_1, 0\}, & q \in Q_1 \\ \{M_2, c\}, & q \in Q_2 \ (q \notin Q_1) \end{cases}$$

i.e., the relation

$$R q' M_1 q = q' M_2 q + cu^2 \qquad (R > 0) \qquad (16)$$

is fulfilled on the boundary of Q_1 and system (1) with control (3) in region Q_1 and control (11) in region Q_2 is strictly optimal in the sense of the criterion

$$J_\Sigma = \sum_{\nu=1}^{2} \omega_\nu (q, t) J_\nu, \quad \sum_{\nu=1}^{2} \omega_\nu = 1, \quad \omega_1 (q, t) = \begin{cases} 1, & q \in Q_1 \\ 0, & q \in Q_2 (q \notin Q_1) \end{cases}$$

A criterion of form (15) satisfies all the requirements listed earlier. Thus, the requirement of simplicity is fulfilled because to seek for the coefficients $k_i^{(1)}$, $k_i^{(2)}$ in controls (3) and (11) we need one and the same procedure for solving Riccati algebraic equations. The condition $|u_*| < 1$ is usually fulfilled in region Q_1. Therefore; instead of condition (16) there are fulfilled the conditions

$$RM_1 = M_2 + ck'k, \quad k = (k_1^{(2)}, \ldots, k_n^{(2)}) \qquad (17)$$

System (5), (7) - (9), (13), (17) establishes the algorithmic connection between the known and the unknown coefficients and defines the set of optimality criteria of form (15) for system (1), as well as the boundaries for the switching of control (15) to control (3). The solution of the system obtained can be found by one of the methods considered in [11].

As an example we consider the system

$$q_1^{\cdot} = q_2, \quad q_2^{\cdot} = q_3, \quad q_3^{\cdot} = -2q_1 - 2q_2 - 5q_3 + u, \quad |u| \leqslant 1, \quad q(0) = q_0 \qquad (18)$$

The combined control

$$u = \begin{cases} \operatorname{sat}(-10q_1 - 19.5q_2 - 5.5q_3) & q \not\equiv Q_1 \\ -\operatorname{sign}(1.41q_1 + 3q_2 + q_3), & q \in Q_1 \end{cases} \tag{19}$$

satisfies the specifications imposed on the system's transient response concerning time-optimality in region Q_2 and accuracy and small sensitivity in region Q_1. Region Q_1, found from relations (7) - (9), has the form

$$|0.203\, q_1 + q_2| \leqslant 0.033, \quad |2.01\, q_1 + q_2| \leqslant 0.192, \quad |2\, q_1 + q_2 + q_3| \leqslant 0.033$$

The optimizing functional (15) for system (18) and the combined control (19) are defined by the following parameters:

$$M_1 = \begin{vmatrix} 2 & 1.62 & 0.46 \\ 1.62 & 7.10 & 0.89 \\ 0.46 & 0.89 & 1 \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 1.40 & 0 & 0 \\ 0 & 4.71 & 0 \\ 0 & 0 & 0.99 \end{vmatrix}, \quad c = 0.01$$

The graph of the transient response in the phase plane for $q_0 = (0.33, 0.67, 0)$ is shown in Fig. 1 and the graph of the control function, in Fig. 2.

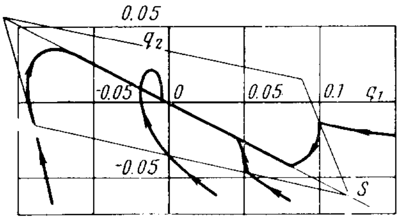

Fig. 1.

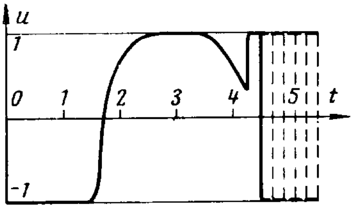

Fig. 2.

Appendix 1. Solution of the inverse optimization problem. Following [9] we write the characteristic equation for the closed system (1), (11) and the Euler-Lagrange equations set up for functional (12)

$$1 + \sum_{i=1}^{n} \gamma_i \lambda^i = 0, \qquad 1 + \frac{1}{\beta_1} \sum_{i=1}^{n} (-1)^i \beta_{i+1} \lambda^{2i} = 0$$

$$\gamma_i = \frac{\alpha_{i+1}}{\alpha_1}, \quad \alpha_i = a_i - k_i^{(2)} \quad (i = 1, \ldots, n), \quad \alpha_{n+1} = 1 \tag{A.1}$$

$$\beta_s = \sum_{i=1}^{n+1} (-1)^{i+s} (m_{i2s-i} c^{-1} + a_i a_{2s-i}) \quad (s = 1, \ldots, n+1)$$

$$m_{ln+1} = m_{n+1l} = 0 \quad (l = 1, \ldots, n+1), \quad a_{n+1} = 1 \tag{A.2}$$

The second equation of (A.1) can be written as

$$\left(1 + \sum_{i=1}^{n} \delta_i \lambda^i\right)\left(1 + \sum_{i=1}^{n} (-1)^i \delta_i \lambda^i\right) = 0 \tag{A.3}$$

where the first factor corresponds to the group of roots λ_s ($s = 1, \ldots, n$) with negative real parts of the second equation of (A.1). By definition, the closed system (1), (11) is asymptotically stable, therefore in order for a functional whose characteristic equation has the form (A.1) to be that functional for which the given control (11) is optimal, it is necessary and sufficient that the condition $\delta_i = \gamma_i$ ($i = 1, \ldots n$) be fulfilled. From (A.2), (A.3) follows

$$\frac{\beta_p}{\beta_1} = \sum_{i=1}^{n+1} (-1)^{i+p} \gamma_i \gamma_{2p-i} \qquad (p = 1, \ldots, n+1) \tag{A.4}$$

Hence, with due regard to (A.1), we find

$$\frac{\beta_{n+1}}{\beta_1} = (-1)^{2n+2} \gamma_{n+1}^2 = \frac{1}{\alpha_1^2} = \frac{1}{\beta_1} = \frac{1}{m_{11}c^{-1} + a_1^2}$$
$$\beta_p = \sum_{i=1}^{n+1} (-1)^{i+p} \alpha_i \alpha_{2p-1} \qquad (p = 1, \ldots, n+1) \tag{A.5}$$

We obtain relation (13) by comparing expressions (A.2) and (A.5).

Let us find the solution of the inverse optimization problem for system (1) with control (3) and optimality criterion (4). It is evident that the sliding segments on S can be treated as the result of the shrinking of an optimality region L of system (1) in the sense of some criterion of form (12) as $c \to 0$. Here the strip with boundaries (14) shrinks to some hyperplane of form (2). The expression for hyperplane S is determined to within a constant factor $R > 0$. Therefore, we consider the ratio $k_i^{(1)} / k_1^{(1)}$ by assuming that

$$\frac{k_i^{(1)}}{k_1^{(1)}} = \lim_{c \to 0} \frac{k_i^{(2)}}{k_1^{(2)}}$$

From expressions (A.2), (A.4) we have

$$\sum_{i=1}^{n+1} (-1)^{i+p} \left[\frac{m_{i2p-i}c^{-1} + a_i a_{2p-i}}{m_{11}c^{-1} + a_1^2} - \frac{a_i - k_i^{(2)}}{(a_1 - k_1^{(2)})^2} (a_{2p-i} - k_{2p-i}^{(2)}) \right] = 0$$
$$k_i^{(2)} = -A_{ni} c^{-1} \qquad (i = 1, \ldots, n) \tag{A.6}$$

where A_{ni} are the coefficients of the optimal Liapunov generating function [7]. We obtain (5) by going to the limit as $c \to 0$ in expression (A.6). The validity of relations (5) can be established by direct verification of the optimality of control law (3) for system (1) in the sliding mode in the sense of criterion (4), using the results of [3].

Appendix 2. Construction of region Q_1. The fulfillment of Condition 2 implies the absence in region Q_1 of trajectories of system (1), (3) intersecting the boundaries of region Q_1. This requirement is expressed analytically by the inequality

$$f(q, u)|_\Gamma \leqslant 0 \tag{A.7}$$

where $f(q, u) = 0$ is the relation defining (together with the expressions for the boundary Γ of region Q_1) the manifold of trajectories tangent to the boundary. In the general case expression (6) defines an $(n-r)$-dimensional strip in an n-dimensional space. For Condition 2 to be fulfilled in region Q_1 it is necessary to impose certain constraints on the matrix Φ of system (1) [12]. Let $r = n$ and let the inequalities

$$|q_i| \leqslant c_i, \quad c_i > 0 \quad (i = 1, \ldots, n) \tag{A.8}$$

hold. The fulfillment of Condition 3 implies that one of the conditions [4]

$$k'\Phi q + k'I \leqslant 0, \quad k'\Phi q + k'I \geqslant 0$$
$$k' = (k_1, \ldots, k_n) = (k_1^{(1)}, \ldots, k_n^{(1)}) \tag{A.9}$$

is fulfilled on the hyperplane $S = 0$ in region Q_1. Thus, for the fulfillment of the conditions imposed on the boundary of region Q_1 it is sufficient to require that relations (A. 7), (A. 8), and one of (A. 9) hold when relations (6) are fulfilled. The connection between the known and the unknown coefficients can be established if we make use of Farkas' theorem [6]: in order that the inequality

$$\varphi(q) \geqslant c \tag{A.10}$$

follow from the joint system of inequalities

$$f_i(q) \geqslant d_i \qquad (i = 1, \ldots, p) \tag{A.11}$$

it is necessary and sufficient that there exist p nonnegative numbers $\lambda_i \geqslant 0$ $(i = 1, \ldots\ p)$ such that the relations

$$\sum_{i=1}^{p} \lambda_i f_i(q) = \varphi(q), \quad \sum_{i=1}^{p} \lambda_i d_i \leqslant c \tag{A.12}$$

are fulfilled. Conditions (A. 9) are fulfilled simultaneously be virtue of the symmetry of region Q_1. Suppose that the validity of the first of inequalities (A. 9) follows from the fulfillment of conditions (2) and

$$-\sum_{j=1}^{n} b_{1j} q_j \geqslant -d_1 \tag{A.13}$$

By substituting the value of q_n found from (2) into expression (A. 13) and into the first expression of (A. 9), we obtain

$$-\sum_{i=1}^{n-1} \left(b_{1i} - b_{1n}\frac{k_i}{k_n}\right) q_i \geqslant -d_1, \quad -\sum_{i=1}^{n-1} \left(k_{i-1} - a_i k_n - \frac{k_{n-1} - a_n k_n}{k_n} k_i\right) q_i \geqslant k_n \tag{A.14}$$

Applying Farkas' theorem (by considering the first inequality of (A. 14) as (A. 10) and the second inequality of (A. 14) as (A. 11)), instead of (A. 12) we have the relation from (9) and the relation

$$\lambda \sum_{i=1}^{n-1} \left(b_{1i} - b_{1n}\frac{k_i}{k_n}\right) q_i = \sum_{i=1}^{n-1} \left(k_{i-1} - a_i k_n - \frac{k_{n-1} - a_n k_n}{k_n} k_i\right) q_i$$

fulfilled, obviously, if the first relation from (9) holds.

By Farkas' theorem the fulfillment of inequalities (A. 8) as a consequence of the system of inequalities (6) implies the fulfillment of the inequalities in (7) and of the following equalities which are fulfilled if the equalities in (7) hold:

$$(\Lambda_s^{(1)} - \Lambda_s^{(2)})' Bq = q_s \qquad (s = 1, \ldots, n)$$

The hypersurface S divides region Q_1 into two subregions Q_1^- $(u = -1)$ and Q_1^+ $(u = +1)$. Because the region Q_1 is symmetric about the origin, requirement (A. 7) is fulfilled simultaneously in the subregions Q_1^- and Q_1^+. Therefore, we consider only the region Q_1^- which is defined by relations (6) and the inequality

$$\sum_{i=1}^{n} k_i^{(1)} q_i \leqslant 0 \tag{A.15}$$

In region Q_1 let the hypersurface S intersect with all the hyperplanes corresponding to inequalities (6). The condition for the tangency of the trajectories of system (1) for

$u = -1$ with the boundaries of region Q_1

$$\pm \sum_{j=1}^{n} b_{sj} q_j = - d_s \qquad (s = 1, \ldots, n) \tag{A.16}$$

has the form

$$\sum_{j=1}^{n} \left(- a_j b_{sn} + b_{sj-1}\right) q_j = b_{sn} \tag{A.17}$$

Further, the upper sign in the expressions corresponds to the "plus" sign in (A.16), the lower, to the "minus" sign. Instead of relation (A.17) we can write the system of inequalities

$$\mp \sum_{j=1}^{n} (-a_j b_{sn} + b_{sj-1}) q_j \geqslant \mp b_{sn} \tag{A.18}$$

Without loss of generality we can set $b_{sn} = 1$. Substituting the values of q_n found from (A.16) into (6), (A.15) and (A.18) we note that when (A.16) is fulfilled, from the system of inequalities

$$\sum_{j=1}^{n-1} (b_{ij} - b_{sj}) q_j \geqslant - d_i \pm d_s, \quad - \sum_{j=1}^{n-1} (b_{ij} - b_{sj}) q_j \geqslant - d_i \mp d_s \tag{A.19}$$

$$\sum_{j=1}^{n-1} (k_n b_{sj} - k_j) q_j \geqslant \mp k_n d_s \qquad (i, s = 1, \ldots, n; i \neq s)$$

there must follow the fulfillment of the inequalities

$$\pm \sum_{j=1}^{n-1} [a_j - b_{sj-1} - (a_n - b_{sn-1}) b_{sj}] q_j \geqslant \mp 1 + (a_n - b_{sn-1}) d_s \tag{A.20}$$

Applying Farkas' theorem to (A.19), (A.20), we find the following relation:

$$\sum_{i=1, i\neq s}^{n} (l_{is}^{(1)} - l_{is}^{(2)}) \sum_{j=1}^{n-1} (b_{ij} - b_{sj}) q_j + l_s \sum_{j=1}^{n-1} (k_n b_{sj} - k_j) q_j =$$

$$= \mp \sum_{j=1}^{n-1} [a_j - b_{sj-1} - (a_n - b_{sn-1}) b_{sj}] q_j$$

$$\sum_{i=1, i\neq s}^{n} [l_{is}^{(1)} (- d_i \pm d_s) + l_{is}^{(2)} (- d_i \mp d_s)] \mp l_s k_n d_s \leqslant 1 + (a_n - b_{sn-1}) d_s.$$

$$l_s,\ l_{is}^{(1)},\ l_{is}^{(2)} \geqslant 0 \quad (i, s = 1, \ldots, n; i \neq s)$$

which is fulfilled if conditions (8) are satisfied.

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Translated by N. H. C.

UDC 531.3

ON THE NORMAL CONFIGURATION OF CONSERVATIVE SYSTEMS

PMM Vol. 36, №1, 1972, pp. 33-42
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(Received January 13, 1971)

We consider the problem of reducing conservative systems to normal coordinates by means of the method of regularizing time transformation widely used in celestial mechanics. Because it is well known that canonic variables have equal validity we can consider the reduction of the systems also to normal momenta. In connection with the reduction mentioned we introduce the concepts of normal and incompletely normal system configurations. We study the existence conditions for normal configurations, proceeding from the structural properties of the Hamiltonian. In particular, we examine these conditions for systems with complete connections, systems with two degrees of freedom, Liouville-type systems, homogeneous systems, systems admitting of a similarity transformation group, systems possessing radial symmetry, and some others.

1. Definitions and Statement of the Problem. We consider a conservative system with k degrees of freedom, moving in a certain force field with energy constant h. The Hamilton-Jacobi equation has the form